Suggested Solution to Quiz 1

Feb 18, 2016

1. (5 points) For each of the following equations, state the order, type and whether it is nonlinear, linear inhomogeneous, or linear homogeneous:

   (a) $\partial_t u - \partial_x^2 u + 1 = 0$

   (b) $\partial_t^2 u - \partial_x^2 u + u^2 = 0$

   (c) $\partial_{xy}^2 u = \sin(4x)$

   (d) $\partial_x^2 u + \partial_{xy}^2 u + \partial_y^2 u = 0$

   Solution:

   (a) 2nd order(0.25points); parabolic equation(0.5points); linear inhomogeneous(0.5points).

   (b) 2nd order(0.25points); hyperbolic equation(0.5points); nonlinear(0.5points)

   (c) 2nd order(0.25points); hyperbolic equation(0.5points); linear inhomogeneous(0.5points).

   (d) 2nd order(0.25points); elliptic equation(0.5points); linear homogeneous(0.5points).

2. (5 points) Solve the equation $\partial_x u + x\partial_y u = 0$ with the following two conditions:

   (a) $u(0, y) = y^2$.

   (b) $u(x, 0) = x^2$

   Solution: The characteristic curves satisfy the ODE:

   $$\frac{dx}{1} = \frac{dy}{x}$$

   Hence the characteristic curves are

   $$y = \frac{1}{2}x^2 + C \quad \text{(1point)}$$

   Therefore, the general solution is

   $$u(x, y) = f(y - \frac{1}{2}x^2) \quad \text{(1point)}$$

   where f is an arbitrary function.

   (a) By $u(0, y) = y^2$, we have $u(0, y) = f(y) = y^2$.Therefore $u(x, y) = (y - \frac{1}{2}x^2)^2$ on $\mathbb{R}^2$.(1 point)

   (b) By $u(x, 0) = x^2$, we have $u(x, 0) = f(-\frac{1}{2}x^2) = x^2$ which implies $f(z) = -2z$. Therefore $u(x, y) = -2(y - \frac{1}{2}x^2) = x^2 - 2y$ (1 point) on the domain $\{(x, y) : y - \frac{1}{2}x^2 \le 0\}$(1 point).On the domain $\{(x, y) : y - \frac{1}{2}x^2 > 0\}$, the solution of $u(x, y)$ can not be determined uniquely.

   Remark: When $u(0, y)$ is given, we can derive the value of u at every point on $\mathbb{R}^2$ along the characteristic curves.

   When $u(x, 0)$ is given, we can just determine the value of u on the domain $\{(x, y) : y - \frac{1}{2}x^2 \le 0\}$ uniquely along the characteristic curves.While on the domain $\{(x, y) : y - \frac{1}{2}x^2 > 0\}$, the solution cannot be determined by the auxiliary condition.